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Draw a cross through the box (☒) if you have NOT written in this booklet

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TOP SCHOLAR

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New Zealand Qualifications Authority

Scholarship 2025
Calculus

Time allowed: Three hours
Total score: 32

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should answer ALL the questions in this booklet.

Pull out Formulae and Tables Booklet S–CALCF from the centre of this booklet.

Show ALL working. Correct answers only will not be sufficient

If you need more room for any answer, use the extra space provided at the back of this booklet.

Check that this booklet has pages 2–24 in the correct order and that none of these pages is blank.

Do not write in any cross-hatched area. This area will be cut off when the booklet is marked.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

QUESTION ONE

(a) Find all the real solutions of the following equation:

$$\sqrt[3]{125^{x^3-5x^2+11x-3}} = \sqrt[4]{5^{4x^2+12}}$$

$$\sqrt[3]{125^{x^3-5x^2+11x-3}} = \sqrt[3]{5^{3(x^3-5x^2+11x-3)}} \qquad (125 = 5^3)$$

$$= 5^{x^3-5x^2+11x-3}$$

$$\sqrt[4]{5^{4x^2+12}} = 5^{x^2+3}$$

$$\Leftrightarrow x^3-5x^2+11x-3 = \cancel{5}\ x^2+3$$

$$\Leftrightarrow x^3-6x^2+11x-6 = (x-1)(x^2-5x+6)$$

$$= (x-1)(x-2)(x-3)$$

$$= 0$$

$$\Leftrightarrow \cancel{x = 1 \text{ or } 2 \text{ or}} \quad x = 1, 2, 3$$

(b) Find all the complex solutions of the following equation:

$$z = 2 - \frac{2+i}{iz}$$

Check $z=-i$
$-i = 2 - \frac{2+i}{1}$ ✓

Check $z=2+i$
$2+i = 2 - \frac{1}{i}$ ✓

$$z = 2 - \frac{2+i}{iz}$$

$$= 2 + \frac{2i-1}{z} \quad \text{(multiply top and bottom by } i\text{)}$$

$$z^2 = 2z + 2i - 1 \quad \text{(multiply by } z\text{)}$$

$$z^2 - 2z + 1 - 2i = 0$$

$$z = \frac{2 \pm \sqrt{4-4(1-2i)}}{2} \quad \text{(quadratic formula)}$$

$$= 1 \pm \sqrt{1-(1-2i)}$$

$$= 1 \pm \sqrt{2i}$$

$(1+i)^2 = 1 + 2i - 1 = 2i$ so $\pm(1+i)$ are the roots of $2i$

$$z = 1 \pm (1+i)$$

$$= -i \text{ or } 2+i$$

(c) How many different complex solutions does the equation $z^{2025} = \overline{z}$ have?

Justify your answer.

Note: as real numbers and purely imaginary numbers are both subsets of the complex numbers, you should consider them also.

$|z^{2025}| = |z|^{2025}$ and $|\overline{z}| = |z|$

so $|z|^{2025} = |z| \iff |z|(|z|^{2024} - 1) = 0$

$|z|$ is a non-negative real, so $|z| = 0$ or 1

If $|z| = 0$, then $z = 0$

If $|z| = 1$

$z = e^{i\theta}$ where $\theta = \arg(z)$, $0 \leq \theta < 2\pi$

$z^{2025} = e^{i(2025\theta)}$

$\overline{z} = e^{i(-\theta)}$

$z^{2025} = \overline{z} \iff e^{i(2025\theta)} = e^{i(-\theta)}$

$\iff 2025\theta = -\theta + 2\pi n$

$\iff \theta = \frac{\pi}{1013} n$

$0 \leq \theta < 2\pi$ so $0 \leq n < 2026$ so there are 2026 values for n and θ

Alternatively, $\overline{z} = \frac{1}{z}$ so $z^{2025} = \frac{1}{z} \iff z^{2026} = 1$

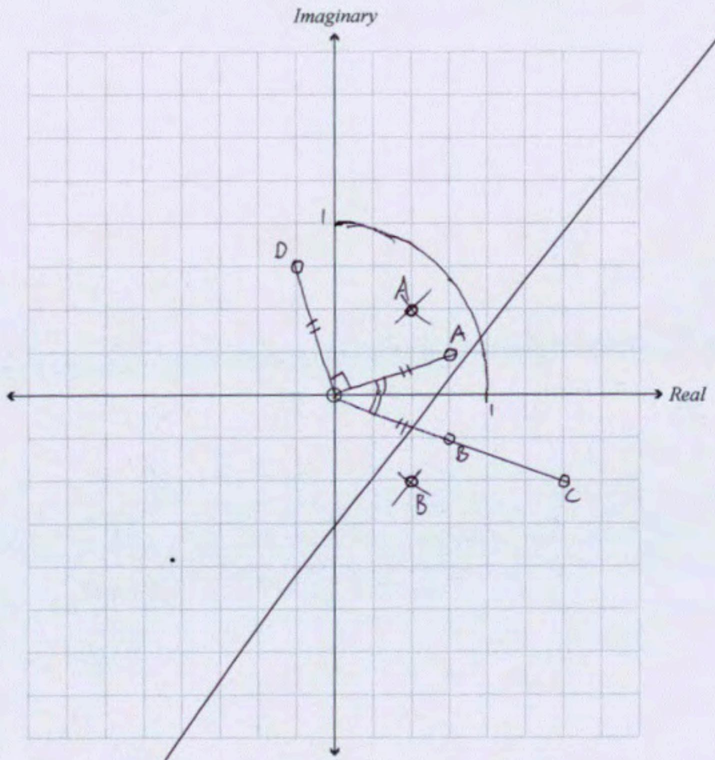
There are 2026 2026^{th} roots of unity.

In total, including $z = 0$, there are 2027 solutions

(d) Let $z = r\operatorname{cis}\theta$ be a complex number with $r > 1$ and $0 < \theta < \frac{\pi}{2}$.
Let the point A represent z in the Argand diagram and the points B, C, and D represent the complex numbers $\overline{z}$, $\frac{1}{z}$, and iz respectively.

If $\angle ABC = 60°$ and $BC = 1.5$, find the area of the quadrilateral ABCD.

Use the Argand diagram below to support your working.



B is the reflection of A over the real axis

A is ~~in~~ strictly inside the first quadrant of the unit circle

D is A rotated by 90° anticlockwise

$\arg\left(\frac{1}{z}\right) = -\arg z$

$= \arg(\bar{z})$

so C lies on ray OB (O is origin)

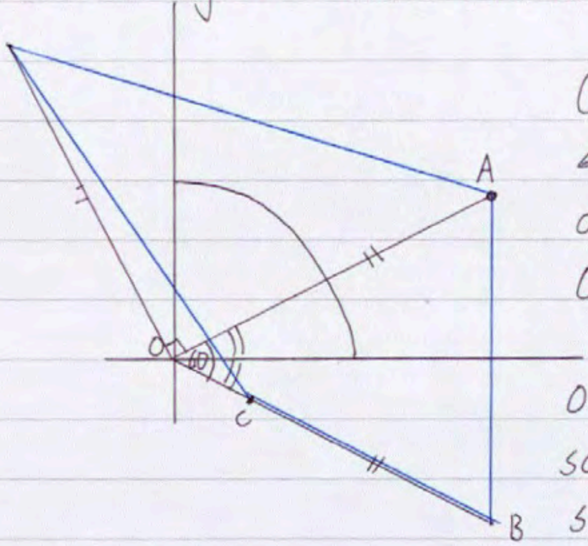
$\left|\frac{1}{z}\right| = \frac{1}{|z|}$

$= \frac{1}{r}$

> 1 (because $0 < r < 1$)

Wait I misread, sorry !!

$0<\theta<\frac{\pi}{2}$ so A is strictly inside the first quadrant (not on the axis)
$r>1$ so A is strictly outside the unit circle
~~B Z~~ $B=\bar{z}$ is the reflection of $A=z$ ~~on~~ over the real axis
$\arg(\frac{1}{\bar{z}})=-\arg(z)=\arg(\bar{z})$ so $C=\frac{1}{\bar{z}}$ lies on line OB (where O is origin)
$|\frac{1}{\bar{z}}|=\frac{1}{|z|}=\frac{1}{r}<1$ because $r>1$, so C lies strictly inside the unit circle.
$D=iz$ is A rotated 90° anticlockwise about origin
Here's a diagram that we will use.



$OA=OB=OD=r$
$\cancel{\angle\theta+}$ $\angle AOD=90°$
$\arg(A)=-\arg(B)=-\arg(C)$
$OC=\frac{1}{r}$
$\angle ABC=60°$
$OA=OB$ so AOB is isosceles
so $\angle OAB=\angle OBA=60°$
so AOB is equilateral and $\angle AOB=60°$

$$\begin{aligned}BC&=OB-OC\\&=r-\frac{1}{r}\\&=1.5\end{aligned}$$

$$\begin{aligned}r^2-1.5r-1&=\frac{1}{2}(2r^2-3r-2)\\&=\frac{1}{2}(r-2)(2r+1)\\&=0\end{aligned}$$

so $r=2$ or $-\frac{1}{2}$ (reject because $r>0$)
$\angle BOD=\cancel{60+9}\ \angle BOA+\angle AOD=60+90=150°$

$$\begin{aligned}[ABCD]&=[ABOD]-[COD]\\&=[ABO]+[ADO]-[COD]\\&=\cancel{\tfrac{1}{2}r^2 A}\ \tfrac{1}{2}r^2\sin 60+\tfrac{1}{2}r^2-\tfrac{1}{2}r\times\tfrac{1}{r}\sin 150 \quad (A=\tfrac{1}{2}ab\sin C)\\&=\tfrac{1}{2}(2)^2\sin 60+\tfrac{1}{2}(2)^2-\tfrac{1}{2}\sin 150 \quad (r=2)\\&=\sqrt{3}+2-\tfrac{1}{4}=\tfrac{7}{4}+\sqrt{3}\end{aligned}$$

QUESTION TWO

(a) A function is defined parametrically by the equations:

$x = 2t + t^2$

$y = p(t)$

where:

- $p(t) = at^3 + bt^2 + ct + d$
- $p(1) = 2.5$
- $p'(1) = 6$

Given that $\frac{d^2y}{dx^2} = \frac{3}{4(1+t)}$, find the value of $p(2)$.

$x(t) = 2t + t^2$

$\cancel{y(t) = at^3 + bt^2 + ct + d}$

$p(t) = at^3 + bt^2 + ct + d = y(t)$

$\boxed{p(1) = a + b + c + d = 2.5 \text{ or } \tfrac{5}{2} \quad (1)}$

$p'(t) = 3at^2 + 2bt + c$

$\boxed{p'(1) = 3a + 2b + c = 6 \quad (2)}$?

$\frac{dx}{dt} = 2 + 2t$

$\frac{dy}{dt} = 3at^2 + 2bt + c$

$\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt}$

$= \frac{3at^2 + 2bt + c}{2t + 2}$

$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{(6at + 2b)(2t + 2) - (3at^2 + 2bt + c)(2)}{(2t+2)^2}$ (Quotient rule)

$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) / \frac{dx}{dt}$

$= \frac{(6at + 2b)(2t + 2) - (3at^2 + 2bt + c)(2)}{(2t+2)^3}$

$= \frac{12at^2 + 12at + 4bt + 4b - 6at^2 - 4bt - 2c}{(2t+2)^3}$

$= \cancel{\frac{t^2(18a) + t(12a + 8b) + (4b + 2c)}{(2t+2)^3}}$

$$= \frac{t^2(9a) + t(6a+4b) + (2b+c)}{4(t+1)^3} \quad \text{(divide top and bottom by 2)}$$

$$= \frac{3}{4(t+1)} \quad \text{(given)}$$

$$\Rightarrow t^2(9a) + t(6a+4b) + (2b+c) = 3(t+1)^2 \quad \text{(multiply both sides by } 4(t+1)^3\text{)}$$

$$= 3t^2 + 6t + 3 \quad \forall t$$

So we can equate coefficients

$9a = 3 \iff a = \frac{1}{3}$

$6a + 4b = 6,$

$= 6\left(\frac{1}{3}\right) + 4b$

$= 2 + 4b \Rightarrow$

so $b = 1$

$2b + c = 2(1) + c$

$= 3$

so $c = 1$

$$\frac{d^2y}{dx^2} = \frac{t^2(6a) + t(12a) + (4b-2c)}{(2t+2)^3} \quad \text{(simplify)}$$

$$= \frac{3at^2 + 6at + 2b - c}{4(t+1)^3} \quad \text{(divide top and bottom by 2)}$$

$$= \frac{3}{4(t+1)} \quad \text{(given)}$$

$$\Rightarrow 3at^2 + 6at + 2b - c = 3(t+1)^2 \quad \text{(multiply both sides by } 4(t+1)^3\text{)}$$

$$= 3t^2 + 6t + 3$$

Equate coefficients: $3a = 3 \Rightarrow a = 1$, $6a = 6 \Rightarrow \boxed{a=1}$, $\boxed{2b - c = 3}$

Substitute $a=1$ into equation(2): $3a + 2b + c = 6 \Rightarrow \boxed{2b + c = 3}$

$2b - c = 3$ and $2b + c = 3$, ~~combim~~ $2c = (2b+c) - (2b-c) = 3 - 3 = 0$ so $\boxed{c = 0}$

~~b~~ $2b + c = 3$ so $\boxed{b = \frac{3}{2}}$, $a + b + c + d = \frac{5}{2}$ so $\boxed{d = 0}$

$P(2) = a \times (2)^3 + b \times (2)^2 + c \times (2) + d = 8a + 4b + 2c + d = 8(1) + 4\left(\frac{3}{2}\right) = 14$

(b) A function f is said to be **odd** if $f(-x) = -f(x)$ for all x in its domain.

Examples of odd functions include $f(x) = x^3$ and $f(x) = \sin(x)$.

Consider an **odd** function f that satisfies the condition $f(1-x) = f(1+x)$ for all real numbers.

Given that $f(1) = 2025$, find the value of $f(1) + f(2) + \ldots + f(2025)$.

$f(1-x) = f(1+x) \quad \forall x \qquad (1)$

f is symmetric about 1

$f(x) = -f(-x) \qquad (2)$

$f(1+x) = f(1-x) \qquad (1)$

$= -f(x-1) \qquad (2)$, f is odd

so $f(x+1) + f(x-1) = 0 \quad \forall x$

shift by 1 to get $f(x+2) + f(x) = 0 \quad \forall x$

Add the two to get $f(x-1) + f(x) + f(x+1) + f(x+2) = 0 \quad \forall x$

So the sum of 4 consecutive values is 0

$2, 3, 4, \ldots, 2025$ is 2024 numbers

2024 is divisible by 4 so $f(2) + f(3) + \cdots + f(2025)$ can be separated into groups of 4 so the sum is 0

Therefore $f(1) + f(2) + \cdots + f(2025) = f(1)$

$= 2025$

It would be nice if we could confirm that such a function exists.

Consider $\cdots = f(-7) = f(-3) = f(1) = f(5) = \cdots = 2025$

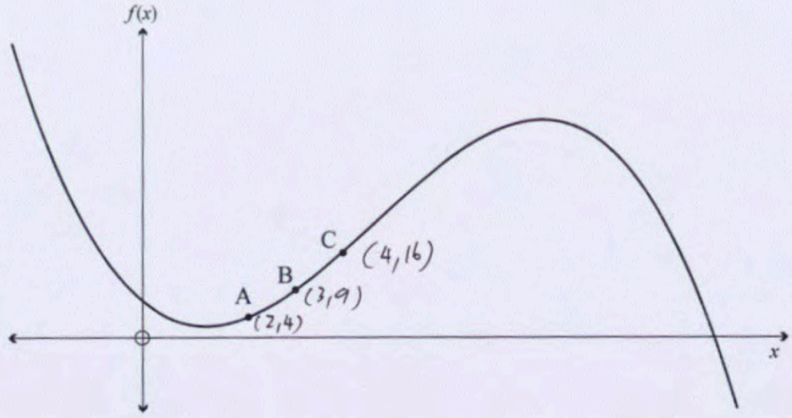
$\cdots = f(-5) = f(-1) = f(3) = f(7) = \cdots = -2025$

$f(\text{everything else}) = 0$

This function is good. :)

(c) Consider the cubic function $f(x) = ax^3 + bx^2 + cx + d$.

The points A(2,4), B(3,9), and C(4,16) all lie on the graph of the cubic, as shown.



Suppose that the lines AB, AC, and BC intersect the cubic again at points D, E, and F, respectively, and that the sum of the x-coordinates of D, E, and F is 25.

Find the y-intercept of the cubic.

Hint: consider the function $g(x) = f(x) - x^2$ in factorised form.

Cubic: $f(x) = ax^3 + bx + cx + d$

We know (2,4), (3,9), (4,16) all lie on the cubic

AB: ~~y~~ gradient $= 9-4 = 5$

$y = 5x - 6$

BC: gradient $= 16-9 = 7$

$y = 7x - 12$

AC: gradient $= \frac{16-4}{4-2} = 6$

$y = 6x - 8$

Let $D = (x_D, y_D)$, $E = (x_E, y_E)$, $F = (x_F, y_F)$

$g(x) = f(x) - x^2$

$g(2) = f(2) - 2^2 = 0$, $g(3) = f(3) - 3^2 = 0$, $g(4) = f(4) - 4^2 = 0$

So by factor theorem, $g(x) = f(x) - x^2$

$= a(x-2)(x-3)(x-4)$

$= ax^3 - 9ax^2 + 26ax - 24a$

~~So $f(x) = ax^3 - 9ax^2 + (26a+1)x - 24a$~~

~~Here's a cool lemma that might work: Let $y=mx+c$ be a line, the sum of x-coordinates of the three intercepts with $f(x)$ will be 9~~

~~Proof: $y=mx+c$ and $y = ax^3-9ax^2+(26a+1)x-24a$~~

~~Equate $\Rightarrow mx+c = ax^3-9ax^2+(26a+1)x-24a$~~

~~$\Rightarrow ax^3-9ax^2+(26a+1$~~

I'm dumb :(

$f(x)= ax^3 + (1-9a)x^2+26ax-24a$

Lemma: let $y=mx+c$ be a line that intersects $f(x)$ at 3 distinct points. The sum of the 3 x-coordinates will be $\frac{9a-1}{a}$

Proof: $y=mx+c = ax^3+(1-9a)x^2+26ax-24a$

$ax^3+(1-9a)x^2+(26a-m)x-24a-c=0$

By Vieta's, sum $= \frac{9a-1}{a}$ or $9-\frac{1}{a}$

Therefore, x-coordinate of $D = 9-\frac{1}{a}-2-3 = 4-\frac{1}{a}$

x-coordinate of $E = 9-\frac{1}{a}-2-4 = 3-\frac{1}{a}$

x-coordinate of $F = 9-\frac{1}{a}-3-4 = 2-\frac{1}{a}$

Sum $= 9-\frac{3}{a} = 25$

so $\frac{3}{a} = -16$

$a = -\frac{3}{16}$

~~$f(x) = -\frac{3}{16}x^3 - 9(-\frac{3}{16})x^2 + (26(-\frac{3}{16})+1)x - 24(-\frac{3}{16})$~~

~~$= -\frac{3}{16}x^3 + \frac{27}{16}x^2 - \frac{31}{8}x + \frac{9}{2}$~~

$f(x) = -\frac{3}{16}x^3 + (1-9(-\frac{3}{16}))x^2 + 26(-\frac{3}{16})x - 24(-\frac{3}{16})$

$= -\frac{3}{16}x^3 + \frac{43}{16}x^2 - \frac{39}{8}x + \frac{9}{2}$

So the constant term is $\frac{9}{2}$

y-intercept

$f(0) = \frac{9}{2}$

QUESTION THREE

(a) Prove that $\frac{1}{1+\sin x} = \sec x(\sec x - \tan x)$.

Hence, or otherwise, find the exact value of:

$$\int_0^{\frac{2\pi}{3}} \frac{1}{1+\sin x}\,dx$$

$\sec x(\sec x - \tan x) = \frac{1}{\cos x}\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$ $\left(\sec = \frac{1}{\cos}, \tan = \frac{\sin}{\cos}\right)$

Assuming $\cos x \neq 0$

$= \frac{1-\sin x}{\cos^2 x}$

$= \frac{1-\sin x}{1-\sin^2 x}$ $(\sin^2 + \cos^2 = 1)$

$= \frac{1-\sin x}{(1-\sin x)(1+\sin x)} = \frac{1}{1+\sin x}$ (difference of squares)

$\frac{d}{dx}(\tan x) = \sec^2 x$ $\quad \frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{+\sin x}{\cos^2 x} = \sec x \tan x$ (quotient)

$\int \frac{1}{1+\sin x}\,dx = \int \sec^2 x - \sec x \tan x\,dx = \tan x - \sec x$

$\int_0^{\frac{2\pi}{3}} \frac{1}{1+\sin x}\,dx = (\tan x - \sec x)\Big|_0^{\frac{2\pi}{3}}$

$= \tan\left(\frac{2\pi}{3}\right) - \sec\left(\frac{2\pi}{3}\right) - \tan 0 + \sec 0$

$= -\sqrt{3} + 2 - 0 + 1 = 3 - \sqrt{3}$

(b) (i) It can be shown that $\tan\left(\frac{\pi}{8}\right) = \sqrt{2} - 1$.

Find a similar expression for $\tan\left(\frac{3\pi}{8}\right)$.

Show all working.

$\tan\left(\frac{3\pi}{8}\right) = \tan\left(\cancel{\frac{2\pi}{16}}\frac{\pi}{4} + \frac{\pi}{8}\right)$

$= \frac{\tan\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{8}\right)}{1 - \tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{8}\right)}$ (Angle-Addition)

$= \frac{1 + (\sqrt{2}-1)}{1 - 1(\sqrt{2}-1)}$ $\left(\tan\frac{\pi}{4} = 1 \text{ and } \tan\frac{\pi}{8} = \sqrt{2}-1\right)$

$= \frac{\sqrt{2}}{2-\sqrt{2}}$

$\cancel{=}\ = \frac{1}{\sqrt{2}-1}$ or $\frac{\sqrt{2}+1}{(\sqrt{2}-1)(\sqrt{2}+1)} = \sqrt{2}+1$

Oh wait there was a cooler way, $\tan\left(\frac{\pi}{2} - x\right) = \frac{\sin\left(\frac{\pi}{2}-x\right)}{\cos\left(\frac{\pi}{2}-x\right)} = \frac{\cos x}{\sin x} = \frac{1}{\tan x}$

$\therefore \tan\left(\frac{3\pi}{8}\right) = \frac{1}{\tan\left(\frac{\pi}{8}\right)} = \frac{1}{\sqrt{2}-1}$

(ii) Hence, or otherwise, find all real solutions of the following system of equations in exact form:

$$\tan\left(\frac{x}{2}\right)\tan y = \sqrt{2} - 1$$

$$\tan\left(\frac{x}{2} + y\right) = 1 + \sqrt{2}$$

$\tan(\frac{x}{2})\tan y = \tan(\frac{\pi}{8}) = \sqrt{2} - 1$

$\tan(\frac{x}{2} + y) = \tan(\frac{3\pi}{8}) = \sqrt{2} + 1$

$= \dfrac{\tan(\frac{x}{2}) + \tan(y)}{1 - \tan(\frac{x}{2})\tan y}$ (Angle-Addition)

$= \dfrac{\tan(\frac{x}{2}) + \tan y}{1 - (\sqrt{2} - 1)}$ $(\tan(\frac{x}{2})\tan y = \sqrt{2} - 1)$

$= \dfrac{\tan(\frac{x}{2}) + \tan y}{2 - \sqrt{2}}$

$\tan(\frac{x}{2}) + \tan y = (2 - \sqrt{2})(\sqrt{2} + 1)$

$= \sqrt{2}(\sqrt{2} - 1)(\sqrt{2} + 1)$

$= \sqrt{2}$

~~tan~~ $\tan(\frac{x}{2})\tan y = \sqrt{2} - 1$ so $\tan(\frac{x}{2}) = \frac{\sqrt{2} - 1}{\tan y}$

Subin $\Rightarrow \frac{\sqrt{2} - 1}{\tan y} + \tan y = \sqrt{2}$

$\tan^2 y - \sqrt{2}\tan y + (\sqrt{2} - 1) = 0$

$\tan y = \dfrac{\sqrt{2} \pm \sqrt{2 - 4(\sqrt{2} - 1)}}{2}$

$= \dfrac{\sqrt{2} \pm \sqrt{6 - 4\sqrt{2}}}{2}$

$\cancel{(\sqrt{2} + 2)^2}$ $(2 - \sqrt{2})^2 = 6 - 4\sqrt{2}$ so $\sqrt{6 - 4\sqrt{2}} = 2 - \sqrt{2}$ so $\tan y = \frac{\sqrt{2} \pm (2 - \sqrt{2})}{2}$

If $\tan y = \frac{\sqrt{2} + (2 - \sqrt{2})}{2} = 1$

$\cancel{y = \tan^{-1}(1)} = y = \frac{\pi}{4} + n\pi$

$\tan(\frac{x}{2}) = \sqrt{2} - 1$ so $\frac{x}{2} = \frac{\pi}{8} + m\pi \Rightarrow x = \frac{\pi}{4} + m2\pi$

Substitute backin andit works

$(x, y) = \cancel{(\frac{\pi}{8} + m\pi, \frac{\pi}{4} + n\pi)}$

$= (\frac{\pi}{4} + m2\pi, \frac{\pi}{4}, n\pi)$

If $\tan y = \frac{\sqrt{2} - (2 - \sqrt{2})}{2} = \frac{2\sqrt{2} - 2}{2} = \sqrt{2} - 1$

$y = \frac{\pi}{8} + n\pi$

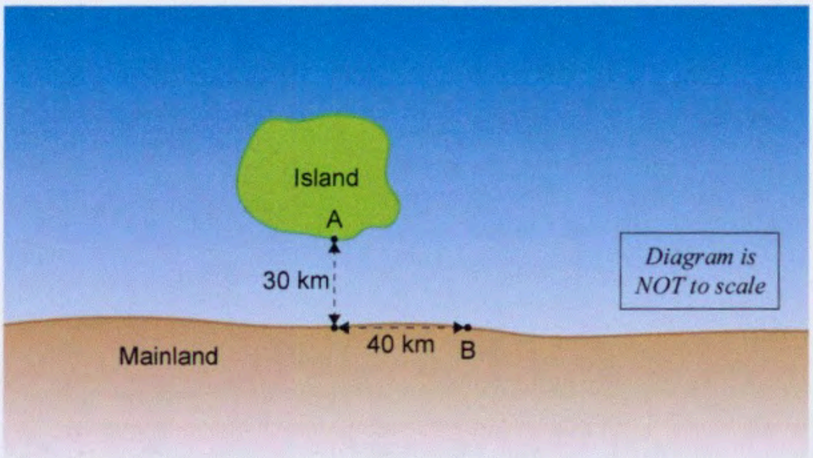
$\tan(\frac{x}{2}) = 1$ so $\frac{x}{2} = \frac{\pi}{4} + m\pi$

$\cancel{x}$ $x = \frac{\pi}{2} + m2\pi$

Sub back in and it works

$(x, y) = (\frac{\pi}{2} + m2\pi, \frac{\pi}{8} + n\pi)$

(c) A local council wishes to supply internet to an island 30 km from the mainland. One option involves running a cable between point A on the island and point B on the mainland, 40 km further down the coast, as shown below.

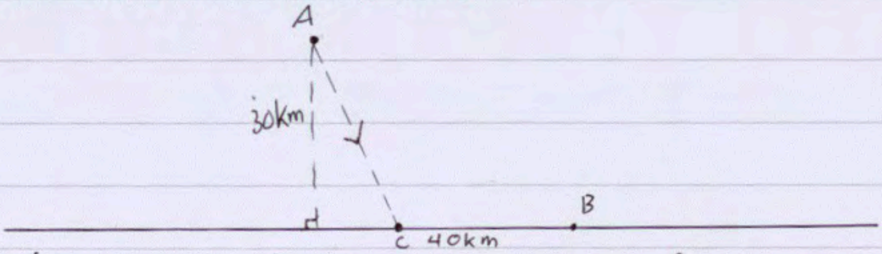


The cost to run a submarine cable under the water is $\$W$ per km, and the cost to run an underground cable on the mainland is $\$L$ per km, where $W > L$.

Use calculus to determine how the cable should be installed to minimise the overall cost of this option for two distinct scenarios:

- $W = 1.5L$
- $W = 1.2L$

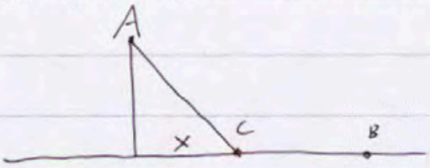
Let's assume that the coast is straight, and that the island is the single point at A (i.e. so this means we will not be ~~using~~ including any underground cables on the island). Now our scenario is like this:



Let C be the point where the cables first reach the land. The cabling from A to C should be a straight submarine cable. Now the optimal cabling from C to B will be a straight underground cable. For a semi-proof of that claim, C to B is a straight line, and submarine cables cost more (W>L) so there is no good reason to go back under water.

We may also assume that C is somewhere between B and the foot ~~fro~~ of the perpendicular from A to the the coast. This is because if C is on the leftside of the foot, we can reflect it over the foot to have the same underwater ~~disland~~ distance and less land distance. If C is on the right of B, moving C to B saves both underwater and land distance.

Let the distance from the foot from A to C be x, $0 \le x \le 40$



$AC = \sqrt{900+x^2}$ (Pythagorean)

$CB = 40-x$

Total cost $= W\cdot AC + L\cdot CB$

$= W\sqrt{900+x^2} + \cancel{W} L\cdot(40-x)$

If $W = 1.5L$

$Cost = 1.5L\sqrt{900+x^2} + L(40-x)$

$= L(1.5\sqrt{900+x^2} + 40 - x)$

$L>0$ so we want to minimise

$y = 1.5\sqrt{900+x^2} + 40 - x$

$\frac{dy}{dx} = \frac{3}{2}\times\frac{1}{2}\times(900+x^2)^{-\frac{1}{2}}\times(2x) - 1$

$= \frac{3}{2}\frac{x}{\sqrt{900+x^2}} - 1$

If $\frac{dy}{dx} = 0$, then $\frac{3x}{2\sqrt{900+x^2}} = 1$

$3x = 2\sqrt{900+x^2}$

$9x^2 = 4(900+x^2)$

$x^2 = 720$

$x = \sqrt{720}$

When $x=0$, $y = 85$

When $x=40$, $y = 75$

When $x=\sqrt{720}$, $y = 40+15\sqrt{5}$

≈ 73.5 (3sf)

So $x=\sqrt{720}$ is optimal

If $W = 1.2L$

$Cost = 1.2L\sqrt{900+x^2} + L(40-x)$

$= L(1.2\sqrt{900+x^2} + 40 - x)$

We want to minimise

$y = 1.2\sqrt{900+x^2} + 40 - x$

$\frac{dy}{dx} = \frac{6}{5}\times\frac{1}{2}(900+x^2)^{-\frac{1}{2}}(2x)\cancel{x} - 1$

$= \frac{6x}{5\sqrt{900+x^2}} - 1$

If $\frac{dy}{dx} = 0$ then $\frac{6x}{5\sqrt{900+x^2}} = 1$

$6x = 5\sqrt{900+x^2}$

$36x^2 = 25(900+x^2)$

$x^2 = \frac{22500}{11}$

$x = \frac{150}{\sqrt{11}} \approx 45.2 > 40$

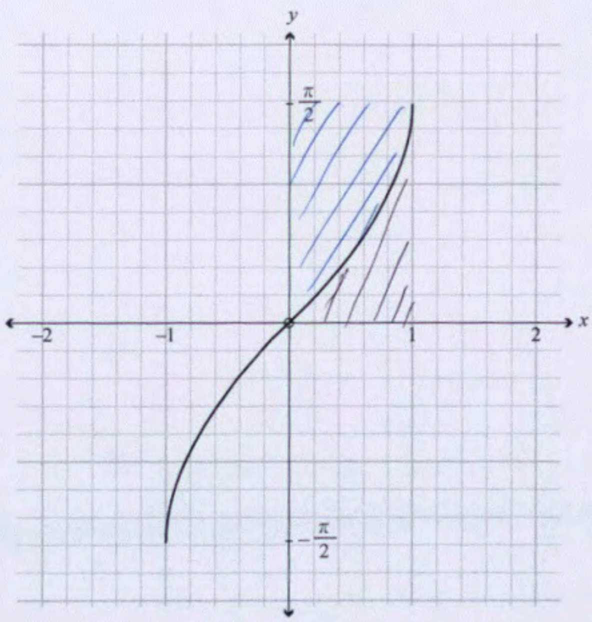
This means y is monotonically increasing or decreasing for $0 \le x \le 40$

When $x=0$, $y = 76$, When $x=40$, $y=60$

So $x=40$ is optimal

QUESTION FOUR

(a) The graph below shows $y = \sin^{-1}(x)$, the inverse sine function, on its domain $-1 \le x \le 1$.



(i) Evaluate the definite integral:

$\int_0^1 \sin^{-1}(x)\,dx$

The blue area in the diagram (the area above $y = \sin^{-1}x$ and below $y = \frac{\pi}{2}$) is $\int_0^{\frac{\pi}{2}} \sin x \, dx = (-\cos x)\Big|_0^{\frac{\pi}{2}}$

$= -\cos\frac{\pi}{2} + \cos 0$

$= 1$

The total area of that rectangle (blue + black) is base × height

$= \frac{\pi}{2}$

So the black area, which is $\int_0^1 \sin^{-1}x \, dx = \frac{\pi}{2} - 1$

(ii) If $x = \sin y$, show that:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Hence, or otherwise, find the minimum value of $g(x) = \sin^{-1}(2x) + \sin^{-1}\left(\frac{\pi}{4} - 2x\right)$.

You may assume that your answer is indeed a minimum.

Implicit differentiation: $\frac{d}{dx}[x] = \frac{d}{dx}[\sin y]$

$1 = \frac{d}{dy}[\sin y]\frac{dy}{dx}$

$= \cos y \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{\cos y}$

$= \frac{1}{\pm\sqrt{1-\sin^2 y}}$

$= \pm\frac{1}{\sqrt{1-x^2}}$

$= \frac{1}{\sqrt{1-x^2}}$ (looking at the graph, $\frac{dy}{dx} > 0$)

I guess a more formal way to prove that would be that $\sin x$ is increasing for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, so therefore $\sin^{-1} x$ is increasing

So $\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$

$\frac{dg}{dx} = \frac{d}{dx}[\sin^{-1}(2x) + \sin^{-1}(\frac{\pi}{4} - 2x)]$

$= \frac{2}{\sqrt{1-4x^2}} + \frac{-2}{\sqrt{1-(\frac{\pi}{4}-2x)^2}} = 2\left(\frac{1}{\sqrt{1-4x^2}} - \frac{1}{\sqrt{1-(\frac{\pi}{4}-2x)^2}}\right)$

Let's first find the domain of g.

$-1 \le 2x \le 1$ so $-\frac{1}{2} \le x \le \frac{1}{2}$ for $\sin^{-1}(2x)$ to exist

$-1 \le \frac{\pi}{4} - 2x \le 1$ so $\frac{\pi}{8} - \frac{1}{2} \le x \le \frac{\pi}{8} + \frac{1}{2}$ for $\sin^{-1}(\frac{\pi}{4} - 2x)$ to exist

so $\frac{\pi}{8} - \frac{1}{2} \le x \le \frac{1}{2}$ is the domain

$g(\frac{1}{2}) = \sin^{-1}(1) + \sin^{-1}(\frac{\pi}{4} - 1) \approx \frac{1}{2}\pi - 0.216$, $g(\frac{\pi}{8} - \frac{1}{2}) = \sin^{-1}(\frac{\pi}{4} - 1) + \sin^{-1}(1) \approx \frac{1}{2}\pi - 0.216$

≈ 1.79

When $\frac{dg}{dx} = 0$, $\sqrt{1-4x^2} = \sqrt{1-(\frac{\pi}{4}-2x)^2} \Rightarrow 1-(2x)^2 = 1-(\frac{\pi}{4}-2x)^2$

$\Rightarrow (2x)^2 = (\frac{\pi}{4} - 2x)^2$

Expand $\Rightarrow 4x^2 = \frac{\pi^2}{16} - \pi x + 4x^2$ so $\pi x = \frac{\pi^2}{16} \Rightarrow x = \frac{\pi}{16}$

When $x = \frac{\pi}{16}$, $g(\frac{\pi}{16}) = \sin^{-1}(\frac{\pi}{8}) + \sin^{-1}(\frac{\pi}{4} - \frac{\pi}{8}) \approx 0.807$ (3sf) or $2\sin^{-1}(\frac{\pi}{8})$

Oh, we probably could've done something cool with Jensen's Inequality.

(b) Consider the function $f(x) = \left(\frac{1}{r^r}\right)^{\frac{1}{r+1}} \cdot x^r$, where r is a positive, real constant, and $x > 0$.

Find the value of r for which $f(x)$ is a solution of the differential equation $f'(x) = f^{-1}(x)$.

Note: if f and f^{-1} are inverse functions, it can be shown that $f\left(f^{-1}(x)\right) = x$ for all x in the domain of f^{-1}.

$\forall$ constant $r \in \mathbb{R}$, $f(x) = \left(\frac{1}{r^r}\right)^{\frac{1}{r+1}} x^r$ for $x>0$

$f'(x) = \frac{d}{dx}\left[\left(\frac{1}{r^r}\right)^{\frac{1}{r+1}} x^r\right]$

$= \left(\frac{1}{r^r}\right)^{\frac{1}{r+1}} \frac{d}{dx}\left[x^r\right]$ (constant rule)

$= \left(\frac{1}{r^r}\right)^{\frac{1}{r+1}} r x^{r-1}$ (power rule)

The $f(x)$ is really just constant $\times x^r$ so the ~~domain~~ range of f is $f(x)>0$ so the domain of f^{-1} is $x>0$.

$f(f^{-1}(x)) = \left(\frac{1}{r^r}\right)^{\frac{1}{r+1}} (f^{-1}(x))^r = x \quad \forall x>0$

$\Rightarrow (f^{-1}(x))^r = (r^r)^{\frac{1}{r+1}} x$

$\Rightarrow f^{-1}(x) = (r)^{\frac{1}{r+1}} x^{\frac{1}{r}} \quad \forall x>0$

$f'(x) = f^{-1}(x) \Rightarrow \left(\frac{1}{r^r}\right)^{\frac{1}{r+1}} r x^{r-1} = r^{\frac{1}{r+1}} x^{\frac{1}{r}}$

We would need $x^{r-1} = x^{\frac{1}{r}}$ so $r-1 = \frac{1}{r}$

$r^2 - r - 1 = 0$

We would also need $\left(\frac{1}{r^r}\right)^{\frac{1}{r+1}} r = r^{\frac{1}{r+1}}$

$\Leftrightarrow \frac{1}{r^r} r^{r+1} = r$ (take both sides to $r+1^{th}$ power)

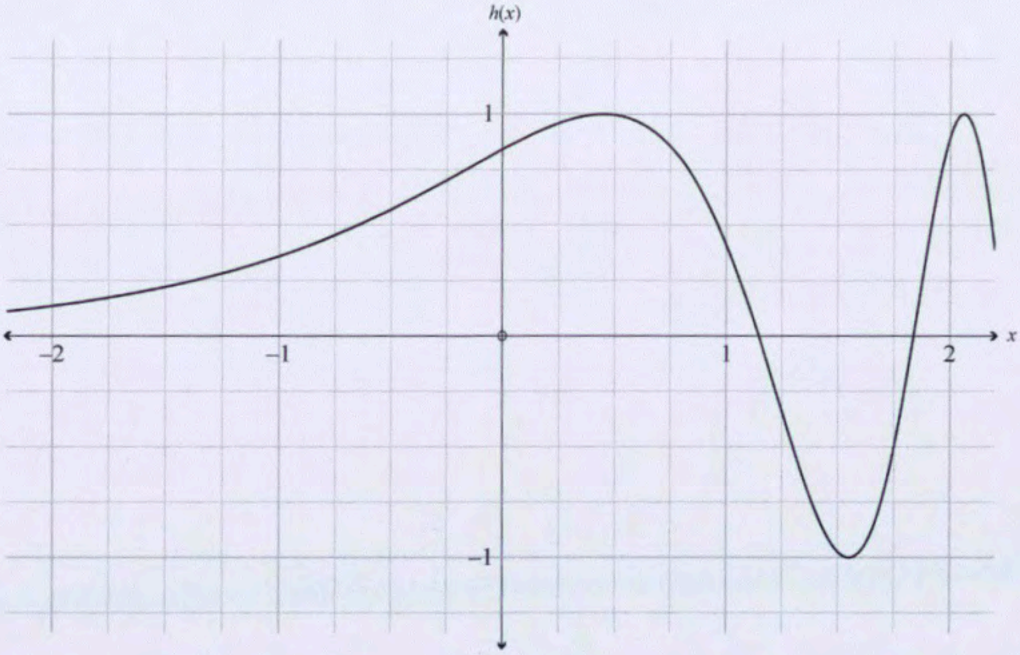
which is always true

The roots to $r^2 - r - 1 = 0$ are $r = \frac{1 \pm \sqrt{1+4}}{2}$

$= \frac{1 \pm \sqrt{5}}{2} = \frac{1+\sqrt{5}}{2}$

$\frac{1-\sqrt{5}}{2} < 0$ but $r>0$ so reject.

(c) The graph below shows part of the graph of $h(x) = \sin(e^x)$.



(i) Find the value of $\lim_{x\to-\infty} h(x)$ and $\lim_{x\to\infty} h(x)$, or state clearly if any do not exist.

Support your answer with mathematical reasoning.

$\lim_{x\to-\infty} h(x) = \lim_{x\to-\infty} \sin(e^{+x})$

$= \lim \sin\left(\lim_{x\to-\infty} e^{+x}\right)$ (sin is continuous)

$= \sin(0) = 0$ (e^x is continuous)

$\lim_{x\to\infty}(h(x))$ does not exist because it will keep oscillating

$h(x) = \sin(e^x) = 1$ if $e^x = 2\pi n + \frac{\pi}{2} \iff x = \ln(2\pi n + \frac{\pi}{2})$

so there are arbitrarily large values of x such that $h(x) = 1$

But $h(x) = -1$ if $e^x = 2\pi n + \frac{3\pi}{2} \iff x = \ln(2\pi n + \frac{3\pi}{2})$ so there are also arbitrarily large values such that $h(x) = -1$. Suppose $\lim_{x\to\infty} h(x) = L$ exists, then $|-L|$ and $|1-L| < \varepsilon$ for arbitrarily small ε, which makes no sense.

(ii) Find the exact value of a that generates the maximum value of the definite integral:

$$I = \int_a^{a+1} \sin(e^x)\,dx$$

You may use the graph above to justify your answer.

Let $u = e^x$, $\frac{du}{dx} = e^x = u$

$\int \sin(e^x)\,dx = \int \sin(u)\frac{1}{u}\,du$

$= \int \frac{1}{u}\sin(u)\,du$

$= \frac{1}{u}(-\cos u) - \int$

$\int \sin(e^x)dx = x\sin(e^x) - \int xe^x\cos(e^x)dx$ (integration by parts

$\frac{d}{dx}(\sin(e^x)) = \cancel{e^x\sin(e^x)}\ e^x\cos(e^x)$

$\frac{d}{dx}(x\sin(e^x)) = \sin(e^x) + xe^x\cos(e^x)$

$\int xe^x\cos(e^x)dx = x(\sin(e^x))$

Maclaurinseries expansions:

$\sin(0)=0, \overset{\cos}{\sin}'(0)=1, \overset{-\sin}{\sin}''(0)=0, \overset{-\cos}{\sin}^{(3)}(0)=-1, \overset{\sin}{\sin}^{(4)}(0)=0$

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$

$\cos(0)=1, \overset{-\sin}{\cos}'(0)=0, \overset{-\cos}{\cos}''(0)=-1, \overset{\sin}{\cos}^{(3)}(0)=0, \overset{\cos}{\cos}^{(4)}(0)=1$

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$

I'm notsure if these work, so I'll do some playing around, feel free to ignore

$\frac{d}{dx}\sin x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots = \cos x$

$\frac{d}{dx}(\cos x) = -\frac{x}{1!} + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots = -\sin x$

$\frac{d}{dx}(\sin(2x)) = \frac{d}{dx}\left(2x - \frac{2^3x^3}{3!} + \frac{2^5x^5}{5!} - \frac{2^7x^7}{7!} + \frac{2^9x^9}{9!} + \cdots\right)$

$= 2 - 2^3\frac{x^2}{2!} + 2^5\frac{x^4}{4!} - 2^7\frac{x^6}{6!} + 2^9\frac{x^8}{8!} + \cdots$

$= 2\left(1 - \frac{2^2x^2}{2!} + \frac{2^4x^4}{4!} + \cdots\right) = 2\cos(2x)$

$\int \cancel{\sin}\cos(x) = \int 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$

$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + C$

$= \sin x + C$

$\int \sin(x) = \int x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

$= \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + C$

$= -\cos x + 1$ Play time over

$\sin(e^x) = e^x - \frac{e^{3x}}{3!} + \frac{e^{5x}}{5!} - \frac{e^{7x}}{7!} + \frac{e^{9x}}{9!} + \cdots$

$\int \sin(e^x)dx = e^x - \frac{e^{3x}}{3\cdot 3!} + \frac{e^{5x}}{5\cdot 5!} - \frac{e^{7x}}{7\cdot 7!} + \frac{e^{9x}}{9\cdot 9!} + \cdots$

$\int_a^{a+1} \sin(e^x)dx = e^{a+1} - \frac{e^{3(a+1)}}{3\cdot 3!} + \frac{e^{5(a+1)}}{5\cdot 5!} - \frac{e^{7(a+1)}}{7\cdot 7!} + \cdots$

$- e^a + \frac{e^{3a}}{3\cdot 3!} - \frac{e^{5a}}{5\cdot 5!} + \frac{e^{7a}}{7\cdot 7!} - \cdots$

$\frac{d}{da}\left[\int_a^{a+1}\sin(e^x)dx\right] = e^{a+1} - \frac{e^{3(a+1)}}{3!} + \frac{e^{5(a+1)}}{5!} - \frac{e^{7(a+1)}}{7!} + \cdots$

$-e^a + \frac{e^{3a}}{3!} - \frac{e^{5a}}{5!} + \frac{e^{7a}}{7!} - \cdots$

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Write the question number(s) if applicable.

QUESTION NUMBER

4 c ii

Maximise $\int_a^{a+1} \sin(e^x)\,dx$

Let $F(x)$ be such that ~~dF~~ $\frac{dF}{dx} = \sin(e^x)$

$\int_a^{a+1} \sin(e^x)\,dx = F(a+1) - F(a)$

$\frac{d}{da}[F(a+1) - F(a)] = \cancel{\sin}\ \sin(e^{a+1}) - \sin(e^a)$

$= 0$

$\boxed{\sin(e^{a+1}) = \sin(e^a)}$

If $e^{a+1} = e^a$

$e = 1$ #

$e^{a+1} = e^a + 2\pi n$ or $\pi - e^a + 2\pi n$

$e^a(e-1) = 2\pi n$ $\qquad$ $\cancel{e^a}\ e^a(e+1) = \pi(2n+1)$

$a = \ln\left(\frac{2\pi n}{e-1}\right)$ or $\cancel{e}$ $\quad a = \ln\left(\frac{\pi(2n+1)}{e+1}\right)$

$n > 0$ $\qquad\qquad$ $n \geq 0$

$a = \ln\left(\frac{2\pi n}{e-1}\right) = 1.30, 1.99, 2.40, \ldots$

$a = \ln\left(\frac{\pi(2n+1)}{e+1}\right) = -0.169, 0.930, 1.44, \ldots$

Looking at the graph, $-0.169 = \ln\left(\frac{\pi}{e+1}\right)$ seems to be when the area under $\sin(e^x)$ is maximised

Extra space if required.
Write the question number(s) if applicable.

QUESTION NUMBER

$$\int_a^{a+1} \sin(e^x)\,dx = e^{a+1} - e^{3(a+1)}/3! + e^{5(a+1)}/5! + \dots$$
$$- e^a + e^{3a}/3! - e^{5a}/5! + \dots$$

If $a = \ln(2\pi$